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Existence of positive periodic solutions for the p-Laplacian system[☆]

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Abstract

In this work, we study the existence of positive periodic solutions for the p-Laplacian system.
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1. Introduction

In this work we consider the following boundary value problem for the p-Laplacian system:

$$(\Phi_p(u'(t)))' + A(t) \cdot u(t) = f(t, u(t)), \quad t \in I = [0, T], \tag{1.1}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{1.2}$$

where $p > 2$, $u = (u_1, u_2, \ldots, u_n)$, $\Phi_p(u) : \mathbb{R}^n \to \mathbb{R}^n$ is given by $(|u|^{p-2}u_1, |u|^{p-2}u_2, \ldots, |u|^{p-2}u_n)$ and $A(t) = (a_{ij}(t))_{n\times n}$ is a matrix-valued function of size $n \times n$.

If $v(t) = \Phi_p(u'(t))$, where $u(t)$ is a positive solution of (1.1) and (1.2), then for any $t \in I$, we have

$$u'(t) = \Phi_p^{-1}(v(t)),$$
$$v'(t) = f(t, u(t)) - A(t)u(t).$$

The system models some phenomena in different physical and other natural sciences: non-Newtonian mechanics, nonlinear elasticity and glaciology, combustion theory, population biology, etc.; see [1–6]. As an example, in the population model, $u(t)$ and $v(t)$ are n-dimensional vectors of the population densities, $\Phi_p^{-1}(v(t))$ and $f(t, u)$ represent some contributions of two kinds of species to each other, while the linear matrix $A(t)$ represents the carrying capacity of the environment of $u(t)$.

Recently, problems concerning periodic solutions for the p-Laplacian have been considered by many authors; see for example [9,10,13]. Most authors study the problems governed by a single equation (see [11,12]) or consider the

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problems for the special case $p = 2$ (see [14–16]). Our consideration is motivated by the results in [1] and [8]. In these papers, the authors studied the following ordinary differential equation with the periodic boundary value condition:

$$\begin{cases} u''(t) + a(t)u(t) = f(t, u(t)), & t \in [0, T], \\ u^{(k)}(0) = u^{(k)}(T), & k = 0, 1, \end{cases} \tag{1.3}$$

where $a(t)$ is a nonnegative T-periodic continuous function, $f(t, u(t))$ is nonnegative continuous and $f(\cdot, u)$ is also a T-periodic function for each $u \in [0, +\infty)$. By applying the fixed point index theory, the authors proved the existence of positive solutions of the problem (1.3). Problem (1.3) corresponds to the special case $n = 1$, $p = 2$ of the problem (1.1) and (1.2).

In this work, we extend the results of [7] and [8]. It is not a simple extension since the dimension $n \geq 2$ and the exponent $p \neq 2$. In one respect, due to $p \neq 2$, the problem is changed into a nonlinear problem, and the well-known Green's function for the linear operator is no longer applicable. On the other hand, the most important aspect is that, unlike for the problem with $n = 1$ and $p = 2$, the upper boundedness of the solution in C^1 norm is needed to prove the existence of the solutions. All these difficulties require us to adopt new methods to solve the problem. Motivated by the ideas in [9], we will make some a priori estimates and use topological degree theory to obtain the existence of positive periodic solutions for (1.1) and (1.2).

This work is organized as follows. In Section 2 we introduce some necessary preliminaries. In Section 3 we give the statement of our main result and its proof.

2. Preliminaries

Assume that $A(t)$ and $f(t, u)$ satisfy the following conditions:

(H1) For every $i, j \in \{1, \ldots, n\}$, $a_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ is a T-periodic continuous function, $A(t)$ is a positive semidefinite matrix and the spectral radius $\rho(A(t))$ is bounded uniformly.
(H2) $f : \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is continuous and $f(\cdot, u) : \mathbb{R}_+ \to \mathbb{R}^n_+$ is also a T-periodic function for each $u \in \mathbb{R}^n_+$.
(H3) There exists a constant $M > 0$, such that for any $|D| > M$, there exists $i \in \{1, 2, \ldots, n\}$, such that $f_i(t, D) > \sum_{j=1}^n a_{ij}(t)D_j$ or $f_i(t, D) < \sum_{j=1}^n a_{ij}(t)D_j$, $t \in I$.

Throughout this work, we denote as $\langle\cdot,\cdot\rangle$ the inner product in $\mathbb{R}^n$. For $n \geq 1$ we set $C = C(I, \mathbb{R}^n_+)$, $C^1 = C^1(I, \mathbb{R}^n_+)$, $C_T = \{u \in C | u(0) = u(T)\}$, $C^1_T = \{u \in C^1 | u(0) = u(T), u'(0) = u'(T)\}$, $L^p = L^p(I, \mathbb{R}^n_+)$ and $W^{2,p} = W^{2,p}(I, \mathbb{R}^n_+)$. The norm in C and C_T will be denoted by $\|\cdot\|_0$, the norm in C^1 and C^1_T by $\|\cdot\|_1$. These norms are defined by

$$\|u\|_0 = \max_{1\leq i\leq n} \|u_i\|_\infty, \qquad \|u\|_1 = \|u\|_0 + \|u'\|_0,$$

where $\|u_i\|_\infty = \sup_{t\in I} |u_i(t)|$. Furthermore, we define the norm in L^p by

$$\|u\|_{L^p} = \left(\sum_{i=1}^n \int_0^T |u_i(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} = \left(\sum_{i=1}^n \|u_i\|^p_{L^p}\right)^{\frac{1}{p}},$$

where $\|u_i\|_{L^p} = \left(\int_0^T |u_i(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}}$.

Definition 2.1. By a positive solution of (1.1) and (1.2), we mean a function $u : I \to \mathbb{R}^n_+$ of class C^1 with $\Phi_p(u')$ absolutely continuous, which satisfies (1.1) and (1.2) a.e. on I.

Now we will introduce a useful lemma, which plays a fundamental role in the proof of the main theorem, and it was proved by Raúl Manásevich and Jean Mawhin in [9].

Consider the periodic boundary problem

$$(\Phi_p(u'))' = f(t, u, u'), \quad t \in I, \tag{2.1}$$

$$u(0) = u(T), \qquad u'(0) = u'(T), \tag{2.2}$$

where the function $f : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be *Carathéodory*.

Lemma 2.1. *Assume that Ω is an open bounded set in C_T^1 such that the following conditions hold.*

(1) *For each $\lambda \in (0, 1)$ the problem*

$$(\Phi_p(u'))' = \lambda f(t, u, u'), \quad u(t) = u(T), \quad u'(t) = u'(T), \tag{2.3}$$

has no solution on $\partial\Omega$.

(2) *The equation*

$$F(a) := \frac{1}{T}\int_0^T f(t, a, 0)\,\mathrm{d}t = 0, \tag{2.4}$$

has no solution on $\partial\Omega \cap \mathbb{R}^n$.

(3) *The Brouwer degree*

$$\deg(F, \Omega \cap \mathbb{R}^n, 0) \neq 0. \tag{2.5}$$

Then problem (2.1) and (2.2) has a solution in $\overline{\Omega}$.

3. Main result and its proof

Let

$$\Omega = \{u \in W^{2,p}(I, \mathbb{R}_+^n); u(0) = u(T), u'(0) = u'(T)\},$$

where $W^{2,p}(I, \mathbb{R}_+^n)$ is the usual *Sobolev* space.

The following theorem is the main result of this work.

Theorem 3.1. *If the assumptions* (H_1), (H_2) *and* (H_3) *hold, then the problem (1.1) and (1.2) admits at least one positive periodic solution $u \in C^1$.*

Proof. Suppose $u \in C^1$ is a solution of the following problem:

$$(|u'(t)|^{p-2}u'(t))' = \lambda(f(t, u(t)) - A(t) \cdot u(t)), \quad t \in I, \tag{3.1}$$

$$u(0) = u(T), \qquad u'(0) = u'(T). \tag{3.2}$$

Next, we will show that u must be uniform C^1 bounded for $\lambda \in [0, 1]$.

By the boundary value conditions, we have

$$-\|u'\|_{L^p}^p = \int_0^T \langle u(t), (\Phi_p(u'(t)))' \rangle\,\mathrm{d}t.$$

Multiplying (3.1) by $u(t)$, and integrating the resulting relation on $[0, T]$, due to the periodicity of $u(t)$, we can deduce

$$\begin{aligned}\|u'\|_p^p &= \lambda\int_0^T \langle A(t)\cdot u(t), u(t)\rangle\,\mathrm{d}t - \lambda\int_0^T \langle f(t, u(t)), u(t)\rangle\,\mathrm{d}t \\ &\leq \lambda\rho\sum_{i=1}^n \|u_i\|_{L^p}^2 T^{1-\frac{2}{p}} \\ &\leq \lambda\rho M_p T^{1-\frac{2}{p}}\|u\|_{L^p}^2 \\ &= \lambda C\|u\|_{L^p}^2,\end{aligned} \tag{3.3}$$

where $\rho = \max_{t\in I} \rho(A(t))$, and M_p is a constant that depends merely on p. In addition, by the periodic boundary value conditions we also have

$$0 = \int_0^T (\Phi_p(u'(t)))'\,\mathrm{d}t = \lambda\int_0^T (f(t, u(t)) - A(t)\cdot u(t))\,\mathrm{d}t. \tag{3.4}$$

So we can confirm that for any $i \in \{1, 2, \ldots, n\}$, there exists $t_{i_0} \in I$, such that $u_i(t_{i_0}) \le M$. Suppose the contrary; then there exists $j \in \{1, 2, \ldots, n\}$ such that $u_j(t) > M$ for any $t \in I$, which implies that $|u(t)| > M$ for any $t \in I$. Recalling the condition (H3), there exists an $i \in \{1, \ldots, n\}$ such that

$$\int_0^T \left(f_i(t, u_1(t), \ldots, u_n(t)) - \sum_{j=1}^n a_{ij}(t) u_j(t) \right) \mathrm{d}t > 0$$

or

$$\int_0^T \left(f_i(t, u_1(t), \ldots, u_n(t)) - \sum_{j=1}^n a_{ij}(t) u_j(t) \right) \mathrm{d}t < 0.$$

This is a contradiction to (3.4). Then $\forall i \in \{1, 2, \ldots, n\}$ we have

$$|u_i(t)| = \left| u_i(t_{i_0}) + \int_{t_{i_0}}^t u_i'(s)\,\mathrm{d}s \right| \le M + T^{\frac{1}{q}} \|u_i'\|_{L^p}, \quad t \in I, \tag{3.5}$$

where q is the conjugate exponent of p. The above inequality implies $\|u_i\|_0 \le M + T^{\frac{1}{q}} \|u_i'\|_{L^p}$. Also by $u_i(t) = u_i(t_{i_0}) + \int_{t_{i_0}}^t u_i'(s)\,\mathrm{d}s$, $i = 1, 2, \ldots, n$ and (3.3), we have

$$\begin{aligned} \|u_i\|_{L^p} &\le \|u_i(t_{i_0})\|_{L^p} + \left\| \int_{t_{i_0}}^t u_i'(s)\,\mathrm{d}s \right\|_{L^p} \\ &\le MT^{\frac{1}{p}} + \left(\int_0^T \left| \int_{t_{i_0}}^t u_i'(s)\,\mathrm{d}s \right|^p \mathrm{d}t \right)^{\frac{1}{p}} \\ &\le MT^{\frac{1}{p}} + \|u_i'\|_{L^p} T \\ &\le MT^{\frac{1}{p}} + T\lambda^{\frac{1}{p}} C^{\frac{1}{p}} \|u\|_{L^p}^{\frac{2}{p}}. \end{aligned}$$

Furthermore, we have

$$\|u\|_{L^p} \le C \left(MT^{\frac{1}{p}} + T\lambda^{\frac{1}{p}} C^{\frac{1}{p}} \|u\|_{L^p}^{\frac{2}{p}} \right). \tag{3.6}$$

Because $p > 2$, the above inequality implies that $\|u\|_{L^p}$ is bounded. Consequently, there exists a constant $M_0 > 0$, such that $\|u\|_{L^p} \le M_0$. Combining this with (3.3) we have

$$\|u'\|_{L^p} \le (\lambda C)^{\frac{1}{p}} \|u\|_{L^p}^{\frac{2}{p}} \le (\lambda C)^{\frac{1}{p}} M_0^{\frac{2}{p}} = M_1,$$

and then by (3.5) we can obtain

$$\|u\|_0 \le M + T^{\frac{1}{q}} M_1 = M_2. \tag{3.7}$$

For any $i \in \{1, \ldots, n\}$, by the boundary condition $u_i(0) = u_i(T)$, it is easy to see that there exists $t_i^* \in I$, such that $u_i'(t_i^*) = 0$. Integrating (3.1) from t_i^* to t, we have

$$\begin{aligned} \left| |u_i'(t)|^{p-2} u_i{}'(t) \right| &\le \left| |u'(t)|^{p-2} u_i{}'(t) \right| \\ &= \lambda \left| \int_{t^*}^t \left(f_i(s, u_1(s), \ldots, u_n(s)) - \sum_{j=1}^n a_{ij}(s) u_j(s) \right) \mathrm{d}s \right| \\ &\le \lambda T \widetilde{M} + \lambda \rho \sqrt{n} \sum_{i=1}^n \left(\int_0^T |u_i(t)|^p\,\mathrm{d}t \right)^{\frac{1}{p}} T^{\frac{1}{q}} \\ &\le M_3, \end{aligned} \tag{3.8}$$

where $\widetilde{M} = \sup_{t \in I, d_k \in [0, M_2]} f_i(t, d_1, \ldots, d_n)$. So from (3.8) we have $|u_i'(t)| \leq |\Phi_q(M_3)|$, that is $\|u'\|_0 \leq \Phi_p^{-1}(M_3)$. Combining this with (3.7), we obtain

$$\|u\|_1 \leq M_2 + \Phi_p^{-1}(M_3) = \overline{M}.$$

That is, $\|u(t)\|_1$ is bounded uniformly.

Let $\Omega_1 = \{u \in \Omega \cap \mathbb{R}^n : F(u) = 0\}$, where

$$F(u) = \frac{1}{T} \int_0^T (f(t, u(t)) - A(t) \cdot u(t))\, \mathrm{d}t. \tag{3.9}$$

We will prove that for any $D \in \Omega_1$, $|D| \leq M$.

For any $D \in \Omega_1$, by $F(D) = 0$ we have

$$\frac{1}{T} \int_0^T (f(t, D) - A(t) \cdot D)\, \mathrm{d}t = 0,$$

that is for every $i \in \{1, \ldots, n\}$,

$$\frac{1}{T} \int_0^T \left(f_i(t, D) - \sum_{j=1}^{n} a_{ij}(t) D_j \right) \mathrm{d}t = 0. \tag{3.10}$$

We can confirm that $|D| \leq M$. Otherwise, by (H3), there exists $i_0 \in \{1, 2, \ldots, n\}$, such that $f_{i_0}(t, D) - \sum_{j=1}^{n} a_{i_0 j}(t) D_j > 0$ or $f_{i_0}(t, D) - \sum_{j=1}^{n} a_{i_0 j}(t) D_j < 0$, which implies that $F_{i_0}(D) < 0$ or $F_{i_0}(D) > 0$. That is a contradiction to (3.10).

Finally we will prove that the condition (3) of Lemma 2.1 is also satisfied.

For any $D \in \Omega \cap \mathbb{R}^n$, recalling (H3), if $|D| > M$, there exists $i_0 \in \{1, \ldots, n\}$, such that

$$f_{i_0}(t, D) - \sum_{j=1}^{n} a_{i_0 j}(t) D_j > 0, \tag{3.11}$$

or

$$f_{i_0}(t, D) - \sum_{j=1}^{n} a_{i_0 j}(t) D_j < 0. \tag{3.12}$$

In the following we assume (3.11). As for the case of (3.12), the proof is similar, so we can omit it. Let

$$\Omega_2 = \{D \in \Omega \cap \mathbb{R}^n : \lambda(D - \xi) + (1 - \lambda) F(D) = 0, \lambda \in [0, 1]\},$$

where $\xi \in \mathbb{R}^n_+$ with $0 < |\xi| < M$. We show that Ω_2 is bounded by M. Suppose the contrary; there exists $D \in \Omega_2$ with $|D| > M$. Hence there exists $i_0 \in \{1, \ldots, n\}$ such that (3.11) holds, which implies that $F_{i_0}(D) > 0$. Furthermore, we have $\lambda(D_{i_0} - \xi_{i_0}) + (1 - \lambda) F_{i_0}(D) > 0$. Obviously, this contradicts the definition of Ω_2. That is for any $D \in \Omega_2$, we always have $|D| \leq M$. From the definition of F, it is easy to see that $F : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is completely continuous. Let

$$h_\lambda(D) = \lambda(D - \xi) + (1 - \lambda) F(D),$$

and define

$$\Omega^* := \{u \in \Omega : \|u\|_1 < \overline{M} + M\}.$$

Then clearly $h_\lambda(\partial \Omega^* \cap \mathbb{R}^n) \neq 0$ for any $\lambda \in [0, 1]$. By virtue of the invariance property of homotopy, we obtain

$$\deg(F, \Omega^* \cap \mathbb{R}^n, 0) = \deg(h_0, \Omega^* \cap \mathbb{R}^n, 0) = \deg(h_1, \Omega^* \cap \mathbb{R}^n, 0) = \deg(I, \Omega^* \cap \mathbb{R}^n, \xi) = 1.$$

Up to now, we have proved that the conditions (1), (2) and (3) in Lemma 2.1 are all satisfied. Therefore, the problem (1.1), (1.2) admits a solution in Ω^* by Lemma 2.1. The proof is completed. □

As a simple example for Theorem 3.1, we consider the system

$$\begin{pmatrix} (|u|^{p-2}u_1)' \\ (|u|^{p-2}u_2)' \end{pmatrix} + \begin{pmatrix} 2+\cos\frac{2\pi}{T}t & 0 \\ \sin\frac{2\pi}{T}t+\cos\frac{2\pi}{T}t & 1-\sin\frac{2\pi}{T}t \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1(t,u) \\ f_2(t,u) \end{pmatrix}, \tag{3.13}$$

$$u_1(0) = u_1(T), \qquad u_1'(0) = u_1'(T), \qquad u_2(0) = u_2(T), \qquad u_2'(0) = u_2'(T), \tag{3.14}$$

where $t \in I$ and

$$\begin{pmatrix} f_1(t,u) \\ f_2(t,u) \end{pmatrix} = \begin{pmatrix} u_1^4 + u_2^4 - u_1^2 - u_2^2 + 2 - \cos\frac{2\pi}{T}t \\ (u_1+1)^2 + (u_2+1)^2 - \sin\frac{2\pi}{T}t \end{pmatrix}.$$

It is easy to see the coefficient matrix satisfies condition (H1) and $f_i(t,u)(i = 1, 2)$ satisfies condition (H2). Notice that $f_i(t,u) \to +\infty$ when $|u| \to +\infty$; thus condition (H3) holds. Hence, by Theorem 3.1, problems (3.13) and (3.14) have positive periodic solutions.

Remark 3.1. Indeed, for the case $\Phi_p(s) = (\Phi_{p_1}(s_1), \ldots, \Phi_{p_n}(s_n))$, with, for each $i = 1, 2, \ldots, n$, $p_i > 2$, and $\Phi_{p_i} : \mathbb{R}_+ \to \mathbb{R}_+$ is the one-dimensional p_i-Laplacian, if (H1), (H2), (H3) hold, then the problem (1.1) and (1.2) admits at least one positive periodic solution.

References

[1] J.I. Diaz, F. de Thelin, On a nonlinear parabolic problem arising in some models related to turbulent flows, SIAM J. Math. Anal. 25 (4) (1994) 1085–1111.
[2] R. Glowinski, J. Rappaz, Approximation of a nonlinear elliptic problem arising in a non-Newtonian fluid flow model in glaciology, Math. Model. Numer. Anal. 37 (1) (2003) 175–186.
[3] U. Janfalk, On certain problem concerning the p-Laplace operator, Likping Studies in Sciences and Technology, Dissertations, 326, 1993.
[4] S. Oruganti, J. Shi, R. Shivaji, Diffusive logistic equation with constant yield harvesting, I: Steady-states, Trans. Amer. Math. Soc. 354 (9) (2002) 3601–3619.
[5] M. Ramaswamy, R. Shivaji, Multiple positive solutions for classes of p-Laplacian equations, Differential Integral Equations 17 (11–12) (2004) 1255–1261.
[6] J.D. Murray, Mathematical Biology, Scientific Publications, Beijing, 1998.
[7] F. Li, Z. Liang, Existence of positive periodic solutions to nonlinear second order differential equations, Appl. Math. Lett. 18 (11) (2005) 1256–1264.
[8] Y. Li, Positive periodic solutions of nonlinear second order differential equations, Acta Math. Sinica 45 (3) (2002) 481–488.
[9] R. Manásevich, J. Mawhin, Periodic solutions for nonlinear systems with p-Laplacian-like operators, J. Differential Equations 145 (1998) 367–393.
[10] P. Yan, Nonresonance for one-dimensional p-Laplacian with regular restoring, J. Math. Anal. Appl 285 (2003) 141–154.
[11] X. Yang, Existence of periodic solutions for quasilinear differential equations, Appl. Math. Comput. 153 (2004) 231–237.
[12] X. Yang, Multiple periodic solutions of a class of p-Laplacian, J. Math. Anal. Appl. 314 (2006) 17–29.
[13] Y. Wang, W. Ge, Existence of periodic solutions for nonlinear differential equations with a p-Laplacian-like operator, Appl. Math. Lett. 19 (2006) 251–259.
[14] J. Henderson, H.B. Thompson, Multiple symmetric positive solutions for a second order boundary value problem, Proc. Amer. Math. Soc. 128 (2000) 2373–2379.
[15] R.I. Avery, Three symmetric positive solutions for a second-order boundary value problem, Appl. Math. Lett. 13 (3) (2000) 1–7.
[16] R.A. Khan, Existence and approximation of solutions of second order nonlinear Neumann problems, Electron. J. Differential Equations 2005 (3) (2005) 1–10.